

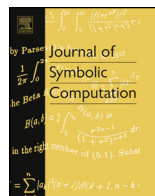


ELSEVIER

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc



On types of degenerate critical points of real polynomial functions

Feng Guo^a, Tiên-Sơn Phạm^b

^a School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, China

^b Department of Mathematics, University of Dalat, 1 Phu Dong Thien Vuong, Dalat, Viet Nam

ARTICLE INFO

Article history:

Received 17 August 2017

Accepted 8 March 2019

Available online 15 March 2019

Dedicated to Professor Do Sang Kim on the occasion of his 65th birthday

MSC:

65K05

68W30

Keywords:

Polynomial functions

Critical points

Degenerate

Types

Tangency varieties

ABSTRACT

In this paper, we consider the problem of identifying the type (local minimizer, maximizer or saddle point) of a given isolated real critical point c , which is degenerate, of a multivariate polynomial function f . To this end, we introduce the definition of faithful radius of c by means of the curve of tangency of f . We show that the type of c can be determined by the global extrema of f over the Euclidean ball centered at c with a faithful radius. We propose algorithms to compute faithful radius of c and determine its type.

© 2019 Elsevier Ltd. All rights reserved.

1. Introduction

Let $f \in \mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$, the polynomial ring over the real number field \mathbb{R} with n variables. Throughout this paper, we denote upper case letters (like X, Y) as variables and lower case letters (like x, y) as points in the ambient spaces. Denote $\mathbf{0}$ as the origin or the vector of zeros. Let c be a point in \mathbb{R}^n such that the gradient $\nabla f(c) = \mathbf{0}$ and the Hessian matrix $\nabla^2 f(c)$ is singular, i.e. c is a degenerate real critical point of f . An interesting problem is to identify the type of c , i.e. is c a local minimizer, maximizer or saddle point of f ? Note that testing whether a given point is a local

E-mail addresses: fguo@dlut.edu.cn (F. Guo), sonpt@du.edu.vn (T.-S. Phạm).

minimizer of a quartic homogeneous polynomial is NP-hard, since this amounts to testing whether the polynomial is nonnegative everywhere, which is an NP-hard problem (Murty and Kabadi, 1987; Nesterov, 2000; Lasserre, 2009).

To solve this problem, it is intuitive to consider the higher order partial derivatives of f at c . However, to the best of our knowledge, it is difficult to obtain a straightforward and simple method, which takes into account the higher order derivatives of f , to systematically solve this problem. When f is a sufficiently smooth function (not necessarily a polynomial), some partial answers to this problem were given in Bolis (1980), Cushing (1975) under certain assumptions on its Taylor expansion at c . When f is a multivariate real polynomial, Qi investigated its critical points and extrema structures in Qi (2004) without giving a computational method to determine their types. Nie gave a numerical method in Nie (2015) to compute all H -minimizers (critical points at which the Hessian matrices are positive semidefinite) of a polynomial by semidefinite relaxations. However, there is no completed procedure in Nie (2015) to verify that a H -minimizer is a saddle point.

Without loss of generality, we suppose that $c = \mathbf{0}$ and $f(\mathbf{0}) = 0$. In this paper, we consider the case when

the origin $\mathbf{0}$ is an isolated real critical point of f ,

i.e., there exists a neighborhood $\mathcal{O} \subseteq \mathbb{R}^n$ of $\mathbf{0}$ such that $\mathbf{0}$ is the only real critical point of f in \mathcal{O} . Note that the occurrence of degenerate critical points is really quite rare. In fact, $f + a_1X_1 + \dots + a_nX_n$ has no degenerate critical point for almost every $(a_1, \dots, a_n) \in \mathbb{R}^n$, see Guillemin and Pollack (1974, Ch. 7). Hence, we aim to present a computational and symbolic method to determine the type of $\mathbf{0}$.

Now let us briefly introduce the basic idea we use to deal with this problem and the contribution made in this paper. Denote \mathbb{R}_+ as the set of positive real numbers and $\|x\|_2$ as the Euclidean norm of $x \in \mathbb{R}^n$. For any $r \in \mathbb{R}_+$, let

$$B_r := \{x \in \mathbb{R}^n \mid \|x\|_2 \leq r\} \quad \text{and} \quad S_r := \{x \in \mathbb{R}^n \mid \|x\|_2 = r\}. \tag{1.1}$$

Define

$$f_r^{\min} := \min\{f(x) \mid x \in B_r\} \quad \text{and} \quad f_r^{\max} := \max\{f(x) \mid x \in B_r\}. \tag{1.2}$$

Obviously, it holds that

- (i) if $\mathbf{0}$ is a local minimizer, then there exists a $r \in \mathbb{R}_+$ such that $f_r^{\max} > 0$ and $f_r^{\min} = 0$;
- (ii) if $\mathbf{0}$ is a local maximizer, then there exists a $r \in \mathbb{R}_+$ such that $f_r^{\max} = 0$ and $f_r^{\min} < 0$;
- (iii) if $\mathbf{0}$ is a saddle point, then for any $r \in \mathbb{R}_+$, it holds that $f_r^{\max} > 0$ and $f_r^{\min} < 0$.

Now we consider the above statements the other way around. That is, can we classify the degenerate critical point $\mathbf{0}$ by the signs of f_r^{\min} and f_r^{\max} ? Two issues have to be addressed.

- (1) If $\mathbf{0}$ is a local minimizer or maximizer, it can be certified by giving a radius r such that $f_r^{\min} = 0$ or $f_r^{\max} = 0$. The difficulty is that if $f_r^{\min} < 0 < f_r^{\max}$ for some $r \in \mathbb{R}_+$, then what is the type of $\mathbf{0}$? Since we do not know if the radius r is sufficiently small, we can not claim that $\mathbf{0}$ is a saddle point. For example, consider the polynomial $f(X_1, X_2) = X_1^2 + (1 - X_1)X_2^4$ (Example 4.18) with $\mathbf{0}$ being an isolated real critical point. Notice that $\mathbf{0}$ is degenerate. If we choose $r = 2\sqrt{2}$, then $f_r^{\min} \leq f(2, 2) = -12 < 0 < 52 = f(-2, 2) \leq f_r^{\max}$. However, if $r < 1$, we have $f(x_1, x_2) > 0$ for any $x \in B_r \setminus \{\mathbf{0}\}$ and $f_r^{\min} = 0$. Hence, $\mathbf{0}$ is not a saddle point but a strict local minimizer. Therefore, in order to determine the type of $\mathbf{0}$ by the global extrema of f over some ball B_r , we need to ensure that the radius r is sufficiently small.
- (2) Note that the optimization problems in (1.2) are themselves NP-hard. In particular, maximizing a cubic polynomial over a unit ball is NP-hard (Nesterov, 2003). Numerically, approximation methods for polynomial optimization problems based on semidefinite relaxations have been extensively studied (Hà and Phạm, 2017; Lasserre, 2009; Laurent, 2009; Marshall, 2008). However,

since we need to certify that f_r^{\min} and f_r^{\max} in cases (i) and (ii) are exactly 0, any numerical errors in the output of approximation methods for (1.2) may mislead us to the wrong case (iii). Symbolically, a univariate polynomial whose roots contain f_r^{\min} and f_r^{\max} may be obtained by means of the KKT system of (1.2) and some elimination computations as in Xiao and Zeng (2015). However, to determine the signs of f_r^{\min} and f_r^{\max} , extra symbolic computations are needed to find one point in some real algebraic set or to certify its emptiness (Safey El Din, 2008; Safey El Din and Schost, 2003).

In this paper, we aim to tackle the above issues and classify isolated critical point $\mathbf{0}$ of f by its global extrema over some Euclidean balls. For the first issue, we define the so-called *faithful radius* (Definition 3.6) of $\mathbf{0}$, via the curve of tangency of f at $\mathbf{0}$ which is studied in Durfee (1998), Hà and Phạm (2008), such that the type of $\mathbf{0}$ can be determined by the signs of f_r^{\max} and f_r^{\min} for any faithful radius r of $\mathbf{0}$ (Theorem 3.7). Provided that an isolation radius (Definition 3.5) of $\mathbf{0}$ is known, we propose an algorithm (Algorithm 4.16) to compute faithful radius of $\mathbf{0}$. We also discuss some strategies to compute isolation radius of $\mathbf{0}$. For the second issue, instead of computing the extrema f_r^{\max} and f_r^{\min} in (1.2), we present an algorithm (Algorithm 5.4) to identify the type of $\mathbf{0}$ by computing isolating intervals for each real root of a zero-dimensional polynomial system, which can be done by algorithms in, for example, Giusti et al. (1995, 2001), Faugère et al. (1993), Alonso et al. (1996), Rouillier (1999).

The paper is organized as follows. Some notation and preliminaries used in this paper are given in Section 2. We define the faithful radius of $\mathbf{0}$ in Section 3. An algorithm for computing a faithful radius of $\mathbf{0}$ is presented in Section 4. We show that how to determine the type of $\mathbf{0}$ in a symbolic way in Section 5. Some conclusion is made in Section 6.

2. Notation and preliminaries

The symbol \mathbb{R} (resp., \mathbb{C}) denotes the set of real (resp., complex) numbers. Denote $\mathbb{R}^{n \times n}$ (resp., $\mathbb{C}^{n \times n}$) as the set of $n \times n$ matrices with real (resp. complex) number entries. $\mathbb{R}[X] = \mathbb{R}[X_1, \dots, X_n]$ denotes the ring of polynomials in variables $X = (X_1, \dots, X_n)$ with real coefficients. Denote $\|X\|_2^2 \in \mathbb{R}[X]$ as the polynomial $X_1^2 + \dots + X_n^2$ in variables X while $\|x\|_2$ as the Euclidean norm of $x \in \mathbb{R}^n$. If f, g are two functions with suitably chosen domains and codomains, then $f \circ g$ denotes the composite function of f and g .

A subset $I \subseteq \mathbb{R}[X]$ is said an ideal if $0 \in I$, $I + I \subseteq I$ and $p \cdot q \in I$ for all $p \in I$ and $q \in \mathbb{R}[X]$. The product of two ideals I and J in $\mathbb{R}[X]$, denoted by $I \cdot J$, is the ideal generated by all products $f \cdot g$ where $f \in I$ and $g \in J$. For $g_1, \dots, g_s \in \mathbb{R}[X]$, denote $\langle g_1, \dots, g_s \rangle$ as the ideal in $\mathbb{R}[X]$ generated by g_i 's, i.e., the set $g_1 \mathbb{R}[X] + \dots + g_s \mathbb{R}[X]$. An ideal is radical if $f^m \in I$ for some integer $m \geq 1$ implies that $f \in I$. The radical of an ideal $I \subseteq \mathbb{R}[X]$, denoted \sqrt{I} , is the set $\{f \in \mathbb{R}[X] \mid f^m \in I \text{ for some integer } m \geq 1\}$. An (resp. real) affine variety is a subset of \mathbb{C}^n (resp. \mathbb{R}^n) that consists of common zeros of a set of polynomials. For an ideal $I \subseteq \mathbb{R}[X]$, denote $\mathbf{V}_{\mathbb{C}}(I)$ and $\mathbf{V}_{\mathbb{R}}(I)$ as the affine varieties defined by I in \mathbb{C}^n and \mathbb{R}^n , respectively. For a polynomial $g \in \mathbb{R}[X]$, respectively replace $\mathbf{V}_{\mathbb{C}}(\langle g \rangle)$ and $\mathbf{V}_{\mathbb{R}}(\langle g \rangle)$ by $\mathbf{V}_{\mathbb{C}}(g)$ and $\mathbf{V}_{\mathbb{R}}(g)$ for simplicity. Given a set $V \subseteq \mathbb{C}^n$, denote $\mathbf{I}(V) \subseteq \mathbb{R}[X]$ as the vanishing ideal of V in $\mathbb{R}[X]$, i.e., the set of all polynomials in $\mathbb{R}[X]$ which equal zero at every point in V . For an ideal $I \subseteq \mathbb{R}[X]$, denote $\dim(I)$ as the Hilbert dimension of I , i.e., the degree of the affine Hilbert polynomial of I . For an ideal $I \subseteq \mathbb{R}[X]$, the decomposition $I = I_1 \cap \dots \cap I_s$ is said the equidimensional decomposition of I if each ideal I_i is pure dimensional, i.e., all its associated primes have the same dimension. For an affine variety $V \subseteq \mathbb{C}^n$, denote $\dim(V) = \dim(\mathbf{I}(V))$ as its dimension. When $\mathbf{V}_{\mathbb{C}}(I)$ is finite, the ideal I is said to be zero-dimensional. For any subset $S \subseteq \mathbb{C}^n$, denote \overline{S}^Z as the Zariski closure of S in \mathbb{C}^n , i.e., $\overline{S}^Z = \mathbf{V}_{\mathbb{C}}(\mathbf{I}(S))$. The l -th elimination ideal I_l of an ideal $I \in \mathbb{R}[X]$ is the ideal of $\mathbb{R}[X_{l+1}, \dots, X_n]$ defined by $I_l = I \cap \mathbb{R}[X_{l+1}, \dots, X_n]$ which can be computed by the Gröbner basis of I with respect to an elimination order of X . For more basic concepts from algebraic geometry, we refer to Cox et al. (2007), Greuel and Pfister (2002). The following procedures are considered as black boxes in this paper (cf. Basu et al., 2006; Greuel and Pfister, 2002):

- Compute the Hilbert dimension of a given ideal $I \subseteq \mathbb{R}[X]$;
- Compute the equidimensional decomposition of a given ideal $I \subseteq \mathbb{R}[X]$, which can be done by, for example, Lecerf (2003);
- Test whether a given ideal $I \subseteq \mathbb{R}[X]$ is radical and compute \sqrt{I} if it is not;
- Compute the vanishing ideal $\mathbf{I}(\sqrt{\mathbf{V}_{\mathbb{C}}(I) \setminus \mathbf{V}_{\mathbb{C}}(J)^Z}) \subseteq \mathbb{R}[X]$ for some ideals $I, J \subseteq \mathbb{R}[X]$;
- Compute isolating intervals for each real root of a zero-dimensional polynomial system. This can be done by algorithms in, for example, Giusti et al. (1995, 2001), Faugère et al. (1993), Alonso et al. (1996), Rouillier (1999).

We recall some background in real algebraic geometry and refer to Bochnak et al. (1998) for more details. A semi-algebraic subset of \mathbb{R}^n is a subset of \mathbb{R}^n satisfying a finite boolean combination of polynomial equations and inequalities with real coefficients. In this paper, \mathbb{R}^n will always be considered with its Euclidean topology, unless stated otherwise. Let $S_1 \subseteq \mathbb{R}^m$ and $S_2 \subseteq \mathbb{R}^n$ be two semi-algebraic sets. A mapping $\psi : S_1 \rightarrow S_2$ is semi-algebraic if its graph is semi-algebraic in \mathbb{R}^{m+n} .

Theorem 2.1. (Bochnak et al., 1998, Theorem 2.3.6) *Every semi-algebraic subset S of \mathbb{R}^n is the disjoint union of a finite number of semi-algebraic sets, each of them semi-algebraically homeomorphic to an open hypercube $(0, 1)^d \subseteq \mathbb{R}^d$ for some $d \in \mathbb{N}$.*

The dimension $\dim(S)$ of a semi-algebraic set $S \subseteq \mathbb{R}^n$ is the maximum of d as in Theorem 2.1. A subset $S \subseteq \mathbb{R}^n$ is connected if for every pair of sets S_1 and S_2 closed in S , disjoint and satisfying $S_1 \cup S_2 = S$, one has $S_1 = S$ or $S_2 = S$. Given points p_1 and p_2 of a subset $S \subseteq \mathbb{R}^n$, a path in S from p_1 to p_2 is a continuous map $\varphi : [a, b] \rightarrow S$ of some closed interval in the real line into S , such that $\varphi(a) = p_1$ and $\varphi(b) = p_2$. A subset $S \subseteq \mathbb{R}^n$ is said to be path connected if every pair of points of S can be joined by a path in S .

Combining Theorems 2.4.4, 2.4.5 and Proposition 2.5.13 in Bochnak et al. (1998), it follows that

Proposition 2.2. *Let S be a semi-algebraic set of \mathbb{R}^n . Then,*

- (i) *the set S has a finite number of connected components which are closed in S ;*
- (ii) *the set S is connected if and only if it is path connected.*

Hence, in the rest of this paper, by saying that a semi-algebraic subset of \mathbb{R}^n is connected, we also mean that it is path connected.

Theorem 2.3. (Milnor, 1968, Curve selection lemma) *Let S be a semi-algebraic subset of \mathbb{R}^n and $x \in \mathbb{R}^n$ a point belonging to the closure of S . Then there exists an analytic semi-algebraic mapping $\varphi : [0, \epsilon] \rightarrow \mathbb{R}^n$ such that $\varphi(0) = x$ and $\varphi((0, \epsilon]) \subset S$.*

3. Faithful radius and types of degenerate critical points

In the rest of this paper, we always denote f as the considered polynomial in $\mathbb{R}[X]$ with $\mathbf{0}$ being an isolated real critical point.

Denote $\text{Crit}_{\mathbb{R}}(f)$ and $\text{Crit}_{\mathbb{C}}(f)$ as the sets of real and complex critical points of f , respectively. Define

$$\Gamma_{\mathbb{R}}(f) := \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \text{ s.t. } \nabla f(x) = \lambda x\}. \tag{3.1}$$

Since $\mathbf{0} \in \text{Crit}_{\mathbb{R}}(f)$, we have

$$\Gamma_{\mathbb{R}}(f) = \left\{ x \in \mathbb{R}^n \mid \frac{\partial f}{\partial X_i} x_j = \frac{\partial f}{\partial X_j} x_i, 1 \leq i < j \leq n \right\}.$$

The real variety $\Gamma_{\mathbb{R}}(f)$ is said the *tangency variety* at the origin Durfee (1998), Hà and Phạm (2008). Geometrically, the tangency variety $\Gamma_{\mathbb{R}}(f)$ consists of all points x in \mathbb{R}^n at which the level set of f is tangent to the sphere in \mathbb{R}^n centered at the origin with radius $\|x\|_2$.

Proposition 3.1. For any $r \in \mathbb{R}_+$ and $u \in B_r$, there exists a point $v \in \Gamma_{\mathbb{R}}(f)$ with $\|v\|_2 \leq \|u\|_2$ such that $f(v) = f(u)$.

Proof. Consider the following optimization problem

$$\min_{x \in \mathbb{R}^n} \|x\|_2^2 \quad \text{s.t. } f(x) = f(u).$$

Since u is a feasible point, there exists a minimizer v with $\|v\|_2 \leq \|u\|_2$. If $\nabla f(v) = \mathbf{0}$, then clearly $v \in \Gamma_{\mathbb{R}}(f)$. Otherwise, the linear independence constraint qualification condition holds at v and therefore v satisfies the Karush–Kuhn–Tucker optimality condition. It implies that $v \in \Gamma_{\mathbb{R}}(f)$. \square

Remark 3.2. For any $r \in \mathbb{R}_+$, the semi-algebraic set $\Gamma_{\mathbb{R}}(f) \cap B_r \setminus \{\mathbf{0}\}$ has finitely many connected components K_i with $\mathbf{0}$ belonging to their closures by Proposition 2.2. For each i , by the curve selection lemma, there exists an analytic curve $\varphi_i : [0, \epsilon] \rightarrow \mathbb{R}^n$ such that $\varphi_i(0) = \mathbf{0}$ and $\varphi_i(t) \in K_i$ for $t \neq 0$. By Proposition 3.1, it can be shown that the behavior of f along the curves φ_i captures all information of f near $\mathbf{0}$. That is, we can identify the type of the critical point $\mathbf{0}$ of f by extremal test of the univariate functions $f \circ \varphi_i$ at 0. This approach was studied in Barone-Netto (1984), Barone-Netto et al. (1996), Hà and Phạm (2008) which, however, provide no general procedures to compute the expressions of the analytic functions φ_i .

For any $r \in \mathbb{R}_+$, recall the definition of f_r^{\min} and f_r^{\max} in (1.2).

Corollary 3.3. For any $r \in \mathbb{R}_+$, we have

$$f_r^{\min} = \min\{f(x) \mid x \in \Gamma_{\mathbb{R}}(f) \cap B_r\} \quad \text{and} \quad f_r^{\max} = \max\{f(x) \mid x \in \Gamma_{\mathbb{R}}(f) \cap B_r\}.$$

Proof. Since f_r^{\min} and f_r^{\max} can be reached, the conclusion follows from Proposition 3.1. \square

Corollary 3.4. The origin $\mathbf{0}$ is not isolated in $\Gamma_{\mathbb{R}}(f)$ and $\dim(\Gamma_{\mathbb{R}}(f) \setminus \text{Crit}_{\mathbb{R}}(f)) \geq 1$.

Proof. Since f is not a zero polynomial, either f_r^{\max} or f_r^{\min} is nonzero for any $r \in \mathbb{R}_+$. Hence by Proposition 3.1, there exists a nonzero $u_r \in \Gamma_{\mathbb{R}}(f) \cap B_r$ such that either $f(u_r) = f_r^{\max} \neq 0$ or $f(u_r) = f_r^{\min} \neq 0$. Thus, $\mathbf{0}$ is not isolated in $\Gamma_{\mathbb{R}}(f)$ since $\lim_{r \rightarrow 0} u_r = \mathbf{0}$.

Note that $\{f(x) \mid x \in \text{Crit}_{\mathbb{R}}(f)\}$ is a finite set by Sard’s theorem (see, for example, Hà and Phạm, 2017, Corollary 1.1). Because $f(u_r) \neq 0$ for each r and $\lim_{r \rightarrow 0} f(u_r) = 0$, there must be infinitely many $u_r \in \Gamma_{\mathbb{R}}(f) \setminus \text{Crit}_{\mathbb{R}}(f)$. Then we have $\dim(\Gamma_{\mathbb{R}}(f) \setminus \text{Crit}_{\mathbb{R}}(f)) \geq 1$ by Theorem 2.1. \square

Definition 3.5. We say that a $R \in \mathbb{R}_+$ is an **isolation radius** of $\mathbf{0}$ if $\text{Crit}_{\mathbb{R}}(f) \cap B_R = \{\mathbf{0}\}$.

Definition 3.6. We say that an isolation radius R of $\mathbf{0}$ is a **faithful radius** if the following conditions hold: (i) $\Gamma_{\mathbb{R}}(f) \cap B_R$ is connected; (ii) $\Gamma_{\mathbb{R}}(f) \cap \mathbf{V}_{\mathbb{R}}(f) \cap B_R = \{\mathbf{0}\}$.

Note that $\Gamma_{\mathbb{R}}(f) \cap B_R$ is also path connected if R is a faithful radius by Proposition 2.2. The following result shows that if R is a faithful radius, then we can classify the degenerate real critical point $\mathbf{0}$ of f by the signs of its global extrema over the ball B_R .

Theorem 3.7. Suppose $R \in \mathbb{R}_+$ is a faithful radius, then

- (1) the origin $\mathbf{0}$ is a local minimizer if and only if $f_R^{\max} > 0$ and $f_R^{\min} = 0$;

- (2) the origin $\mathbf{0}$ is a local maximizer if and only if $f_R^{\max} = 0$ and $f_R^{\min} < 0$;
- (3) the origin $\mathbf{0}$ is a saddle point if and only if $f_R^{\max} > 0$ and $f_R^{\min} < 0$.

Proof. (1) and (2) are clear if we can prove (3). Since $\mathbf{0}$ is an isolated real critical point, we only need to prove the “if” part. By Corollary 3.3, there exists a $u \in \Gamma_{\mathbb{R}}(f) \cap B_R$ such that $f(u) = f_R^{\min}$. Since R is a faithful radius, $\mathbf{0}$ and u are path connected, i.e. there exists a continuous mapping $\phi(t) : [a, b] \rightarrow \Gamma_{\mathbb{R}}(f)$ such that $\mathbf{0} \notin \phi((a, b))$, $\phi(a) = \mathbf{0}$ and $\phi(b) = u$. We have $f(\phi(t)) < 0$ for all $t \in (a, b)$. Otherwise, by continuity, there exists $\bar{t} \in (a, b)$ such that $f(\phi(\bar{t})) = 0$. Since R is faithful, we have $\phi(\bar{t}) = \mathbf{0}$ by the definition, a contradiction. Similarly, let $f_R^{\max} > 0$ be reached at $v \in \Gamma_{\mathbb{R}}(f) \cap B_R$, then there exists a continuous mapping $\varphi(t) : [a, b] \rightarrow \Gamma_{\mathbb{R}}(f)$ such that $\mathbf{0} \notin \varphi((a, b))$, $\varphi(a) = \mathbf{0}$, $\varphi(b) = v$ and $f(\varphi(t)) > 0$ for all $t \in (a, b)$. Therefore, $\mathbf{0}$ is a saddle point of f . \square

There always exists a faithful radius of $\mathbf{0}$. In fact,

Theorem 3.8. *The origin $\mathbf{0}$ is an isolated real critical point of f if and only if there is a faithful radius of $\mathbf{0}$.*

Proof. We only need to prove the “only if” part and assume that $\mathbf{0}$ is an isolated real critical point of f .

(i) By the assumption, there is an isolation radius $R_1 \in \mathbb{R}_+$ such that $\text{Crit}_{\mathbb{R}}(f) \cap B_{R_1} = \{\mathbf{0}\}$.

(ii) Since $\Gamma_{\mathbb{R}}(f) \cap B_{R_1}$ is a closed semi-algebraic set, by Proposition 2.2, it has finitely many connected components C_1, \dots, C_s which are closed in \mathbb{R}^n . Assume that the components C_i , $2 \leq i \leq s$, do not contain $\{\mathbf{0}\}$. For each $2 \leq i \leq s$, since the component C_i is closed and bounded, the function $\sum_{i=1}^n X_i^2$ reaches its minimum on C_i at a minimizer $u^{(i)} \in C_i$. Fix a $R_2 \in \mathbb{R}_+$ such that $0 < R_2 < \min_{2 \leq i \leq s} \|u^{(i)}\|_2$, then $\Gamma_{\mathbb{R}}(f) \cap B_{R_2}$ is connected.

(iii) We claim that there exists $R_3 \in \mathbb{R}_+$ such that $\Gamma_{\mathbb{R}}(f) \cap \mathbf{V}_{\mathbb{R}}(f) \cap B_{R_3} = \{\mathbf{0}\}$. Suppose to the contrary that such R_3 does not exist. By the Curve Selection Lemma, we can find an analytic curve $\phi : [0, \epsilon] \rightarrow \mathbb{R}^n$ such that $\phi(0) = \mathbf{0}$, $f(\phi(t)) = 0$ and $\phi(t) \in \Gamma_{\mathbb{R}}(f) \setminus \{\mathbf{0}\}$ for all $t \in (0, \epsilon]$. For each t , by the definition, $\nabla f(\phi(t)) = \lambda(t)\phi(t)$ for some $\lambda(t) \in \mathbb{R}$ and furthermore,

$$0 = \frac{d(f \circ \phi)(t)}{dt} = \left\langle \nabla f(\phi(t)), \frac{d\phi(t)}{dt} \right\rangle = \frac{\lambda(t)}{2} \frac{d\|\phi(t)\|_2^2}{dt}.$$

By the monotonicity lemma (Hà and Pham, 2017, Theorem 1.8), $\lambda(t) = 0$ and hence $\nabla f(\phi(t)) = \mathbf{0}$ for $0 \leq t \ll 1$, which is a contradiction since $\mathbf{0}$ is an isolated real critical point of f .

Clearly, $R := \min\{R_2, R_3\}$ is a faithful radius of f . \square

4. Computational aspects of faithful radius

In this section, we present some computational criteria and an algorithm for computing a faithful radius of the isolated real critical point $\mathbf{0}$ of the polynomial f .

4.1. Curve of tangency

We now recall some background about the tangency variety at a general point which is studied in Durfee (1998), Hà and Pham (2008). For any $a \in \mathbb{R}^n$, let

$$\Gamma_{\mathbb{R}}(f, a) := \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \text{ s.t. } \nabla f(x) = \lambda(x - a)\}. \tag{4.1}$$

In particular, $\Gamma_{\mathbb{R}}(f) = \Gamma_{\mathbb{R}}(f, \mathbf{0})$. Geometrically, the tangency variety $\Gamma_{\mathbb{R}}(f, a)$ consists of all points x in \mathbb{R}^n at which the level set of f is tangent to the sphere in \mathbb{R}^n centered in a with radius $\|x - a\|_2$.

Proposition 4.1. (Hà and Pham, 2008, Lemma 2.1) *It holds that*

- (i) the set $\Gamma_{\mathbb{R}}(f, a)$ is nonempty, unbounded and a semi-algebraic set;

(ii) there exists a proper algebraic set $\Omega \subset \mathbb{R}^n$ such that for each $a \in \mathbb{R} \setminus \Omega$, the set $\Gamma_{\mathbb{R}}(f, a) \setminus \text{Crit}_{\mathbb{R}}(f)$ is a one-dimensional submanifold of \mathbb{R}^n .

Therefore, $\Gamma_{\mathbb{R}}(f, a)$ is also said *curve of tangency*. Note that for the given $f \in \mathbb{R}[X]$, $\mathbf{0}$ might not belong to Ω as in Proposition 4.1 and then the statement (ii) in Proposition 4.1 is not necessarily true for $\Gamma_{\mathbb{R}}(f) = \Gamma_{\mathbb{R}}(f, \mathbf{0})$. However, in the following we will show that $\Gamma_{\mathbb{R}}(f) \setminus \text{Crit}_{\mathbb{R}}(f)$ is indeed a one-dimensional semi-algebraic set of \mathbb{R}^n after a generic linear change of the coordinates of f .

For $f \in \mathbb{R}[X]$ and an invertible matrix $A \in \mathbb{R}^{n \times n}$, denote $f^A = f(Ax)$ the polynomial obtained by applying the change of variables A to f . Denote $\mathcal{S}^{n \times n} \subset \mathbb{R}^{n \times n}$ (resp. $\mathcal{S}_{++}^{n \times n} \subset \mathbb{R}^{n \times n}$) as the set of symmetric (resp. positive definite) matrices with real number entries. For any matrix $P \in \mathbb{R}^{n \times n}$, define

$$\Gamma_{\mathbb{R}}(f, P) = \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R}, \text{ s.t. } \nabla f(x) = \lambda Px\}.$$

Given an invertible matrix $A \in \mathbb{R}^{n \times n}$ and a subset $S \subseteq \mathbb{R}^n$, let

$$A(S) := \{Ax \mid x \in S\}.$$

Lemma 4.2. *Given an invertible matrix $A \in \mathbb{R}^{n \times n}$, let $P = A^{-T}A^{-1}$, then we have $\Gamma_{\mathbb{R}}(f^A) = A^{-1}(\Gamma_{\mathbb{R}}(f, P))$ and $\text{Crit}_{\mathbb{R}}(f^A) = A^{-1}(\text{Crit}_{\mathbb{R}}(f))$.*

Proof. By the definition, we have

$$\begin{aligned} \Gamma_{\mathbb{R}}(f^A) &= \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \text{ s.t. } \nabla f^A(x) = \lambda x\} \\ &= \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \text{ s.t. } A^T \nabla f(Ax) = \lambda x\} \\ &= \{A^{-1}y \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \text{ s.t. } \nabla f(y) = \lambda A^{-T}A^{-1}y\} \\ &= A^{-1}(\Gamma_{\mathbb{R}}(f, P)). \end{aligned}$$

Similarly, it holds that $\text{Crit}_{\mathbb{R}}(f^A) = A^{-1}(\text{Crit}_{\mathbb{R}}(f))$. \square

Theorem 4.3. *There exists a non-empty Zariski open set $\mathcal{U} \subset \mathbb{R}^{n \times n}$ such that for all $A \in \mathcal{U}$, the set $\Gamma_{\mathbb{R}}(f^A) \setminus \text{Crit}_{\mathbb{R}}(f^A)$ is a one-dimensional semi-algebraic set of \mathbb{R}^n .*

Proof. Clearly, $\mathcal{S}_{++}^{n \times n}$ is an open semi-algebraic subset of $\mathcal{S}^{n \times n} \cong \mathbb{R}^{\frac{n(n+1)}{2}}$, where we identify $P := (p_{ij})_{n \times n} \in \mathcal{S}^{n \times n}$ with

$$(p_{11}, \dots, p_{1n}, p_{22}, \dots, p_{2n}, \dots, p_{nn}) \in \mathbb{R}^{\frac{n(n+1)}{2}}.$$

We first show that $\Gamma_{\mathbb{R}}(f, P) \setminus \text{Crit}_{\mathbb{R}}(f)$ is a semi-algebraic set of dimension ≤ 1 for almost every $P \in \mathcal{S}_{++}^{n \times n}$. To do this, we consider the semi-algebraic map

$$\begin{aligned} F : (\mathbb{R}^n \setminus \text{Crit}_{\mathbb{R}}(f)) \times \mathbb{R} \times \mathcal{S}_{++}^{n \times n} &\rightarrow \mathbb{R}^n \\ (x, \lambda, P) &\mapsto \nabla f(x) - \lambda Px. \end{aligned}$$

We will show that $\mathbf{0} \in \mathbb{R}^n$ is a regular value of the map F . Take any $(x, \lambda, P) \in F^{-1}(\mathbf{0})$, then $x \neq \mathbf{0}$ and $\lambda \neq 0$. Otherwise, we have $\nabla f(x) = \mathbf{0}$ and hence $x \in \text{Crit}_{\mathbb{R}}(f)$, a contradiction. Without loss of generality, we assume that $x_1 \neq 0$. Note that $p_{ij} = p_{ji}$. Then, a direct computation shows that the Jacobian matrix $\text{Jac}(F)$ of the map F contains the following columns

$$-\lambda \cdot \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ 0 & x_1 & 0 & \cdots & 0 \\ 0 & 0 & x_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x_1 \end{bmatrix},$$

which correspond to the partial derivatives of F with respect to the variables p_{1j} for $j = 1, \dots, n$. Therefore, for all $(x, \lambda, P) \in F^{-1}(\mathbf{0})$, we have $\text{rank}(\text{Jac}(F)) = n$ and hence $\mathbf{0}$ is a regular value of F . By Thom's weak transversality theorem (Guillemin and Pollaek, 1974; Hà and Phạm, 2017, Theorem 1.10), there exists a semi-algebraic set $\Sigma \subset \mathcal{S}_{++}^{n \times n}$ of dimension $< \frac{n(n+1)}{2}$ such that for all $P \in \mathcal{S}_{++}^{n \times n} \setminus \Sigma$, $\mathbf{0}$ is a regular value of the map

$$F_P : (\mathbb{R}^n \setminus \text{Crit}_{\mathbb{R}}(f)) \times \mathbb{R} \rightarrow \mathbb{R}^n$$

$$(x, \lambda) \mapsto F(x, \lambda, P).$$

Thus, $F_P^{-1}(\mathbf{0})$ is either empty or a one-dimensional submanifold of \mathbb{R}^n . Since $\Gamma_{\mathbb{R}}(f, P) \setminus \text{Crit}_{\mathbb{R}}(f)$ is the projection of $F_P^{-1}(\mathbf{0})$ on the first n coordinates, by Bochnak et al. (1998, Proposition 2.8.6), we have $\dim(\Gamma_{\mathbb{R}}(f, P) \setminus \text{Crit}_{\mathbb{R}}(f)) \leq 1$.

Next, it is easy to see that

$$\mathcal{S}_{++}^{n \times n} \rightarrow \mathcal{S}_{++}^{n \times n}, \quad P \mapsto P^{-1},$$

is a semi-algebraic homeomorphism. Hence,

$$\Sigma^{-1} := \{P^{-1} \in \mathcal{S}_{++}^{n \times n} \mid P \in \Sigma\} \subset \mathbb{R}^{\frac{n(n+1)}{2}}$$

is a semi-algebraic set of dimension $< \frac{n(n+1)}{2}$. Consequently, by Hà and Phạm (2017, Lemma 1.4), there exists a non-constant polynomial $\mathcal{F} : \mathbb{R}^{\frac{n(n+1)}{2}} \rightarrow \mathbb{R}$ such that

$$\Sigma^{-1} \subset \{Q \in \mathcal{S}_{++}^{n \times n} \mid \mathcal{F}(Q) = 0\}.$$

Note that the corresponding

$$\mathbb{R}^{n \times n} \rightarrow \mathcal{S}^{n \times n}, \quad A \mapsto AA^T,$$

is a polynomial map. Thus,

$$\{A \in \mathbb{R}^{n \times n} \mid \mathcal{F}(AA^T) = 0\}$$

is an algebraic set in $\mathbb{R}^{n \times n}$. Let $\mathcal{I}^{n \times n}$ be the set of all invertible $n \times n$ matrices, which is a non-empty Zariski open set in $\mathbb{R}^{n \times n}$. It follows that $\mathcal{U} := \{A \in \mathcal{I}^{n \times n} \mid \mathcal{F}(AA^T) \neq 0\}$ is a non-empty Zariski open set in $\mathbb{R}^{n \times n}$. Furthermore, by the definition, for all $A \in \mathcal{U}$, we have $P := (AA^T)^{-1} \in \mathcal{S}_{++}^{n \times n} \setminus \Sigma$ and hence, $\dim(\Gamma_{\mathbb{R}}(f^A) \setminus \text{Crit}_{\mathbb{R}}(f^A)) \leq 1$ by Lemma 4.2. Since $A \in \mathcal{U}$ is invertible, we have $\dim(\Gamma_{\mathbb{R}}(f^A) \setminus \text{Crit}_{\mathbb{R}}(f^A)) \geq 1$ by Corollary 3.4 and then the conclusion follows. \square

Remark 4.4. (i) Thom's weak transversality result is used previously in Bank et al. (1997) and also for roadmap computations in Safey El Din and Schost (2017).

(ii) The technique of a generic linear change of variables was also used in Safey El Din and Schost (2003) to show the dimension of polar varieties.

(iii) We may also use the new inner product $\langle x, x' \rangle_P := \langle Px, Px' \rangle$ (and the corresponding norm $\sqrt{\langle x, x' \rangle_P}$) for some generic $P \in \mathcal{S}_{++}^{n \times n}$ instead of using a generic linear change of variables $y = Ax$. In fact, it is not hard to see that with this new inner product, $\Gamma_{\mathbb{R}}(f) \setminus \text{Crit}_{\mathbb{R}}(f)$ is also a one-dimensional semi-algebraic set of \mathbb{R}^n .

We illustrate the result in Theorem 4.3 by the following simple example.

Example 4.5. Consider the polynomial $f(X_1, X_2) = X_1^2 + X_2^2$. We have $\Gamma_{\mathbb{R}}(f) = \mathbb{R}^2$. However, if we make a linear change of variables and let $f^A = (a_{1,1}X_1 + a_{1,2}X_2)^2 + (a_{2,1}X_1 + a_{2,2}X_2)^2$, then $\Gamma_{\mathbb{R}}(f^A) = \{(x_1, x_2) \in \mathbb{R}^2 \mid (a_{1,1}a_{1,2} + a_{2,1}a_{2,2})(x_2^2 - x_1^2) + (a_{1,1}^2 + a_{2,1}^2 - a_{1,2}^2 - a_{2,2}^2)x_1x_2 = 0\}$. Clearly, $\dim(\Gamma_{\mathbb{R}}(f^A) \setminus \text{Crit}_{\mathbb{R}}(f^A)) = 1$ whenever $a_{1,1}a_{1,2} + a_{2,1}a_{2,2} \neq 0$ or $a_{1,1}^2 + a_{2,1}^2 - a_{1,2}^2 - a_{2,2}^2 \neq 0$.

For $f \in \mathbb{R}[X]$ and any matrix $P \in \mathbb{R}^{n \times n}$, let

$$\begin{aligned} \Gamma_{\mathbb{C}}(f) &:= \{x \in \mathbb{C}^n \mid \exists \lambda \in \mathbb{C} \text{ s.t. } \nabla f(x) = \lambda x\}, \\ \Gamma_{\mathbb{C}}(f, P) &:= \{x \in \mathbb{C}^n \mid \exists \lambda \in \mathbb{C} \text{ s.t. } \nabla f(x) = \lambda Px\}. \end{aligned} \tag{4.2}$$

Recall that $\text{Crit}_{\mathbb{C}}(f)$ and $\text{Crit}_{\mathbb{C}}(f^A)$ denote the sets of complex critical points of f and f^A , respectively. As in Lemma 4.2, we still have

Lemma 4.6. *Given an invertible matrix $A \in \mathbb{R}^{n \times n}$, let $P = A^{-T} A^{-1}$, then we have $\Gamma_{\mathbb{C}}(f^A) = A^{-1}(\Gamma_{\mathbb{C}}(f, P))$ and $\text{Crit}_{\mathbb{C}}(f^A) = A^{-1}(\text{Crit}_{\mathbb{C}}(f))$.*

By using the algebraic version of Thom’s weak transversality theorem (cf. Demazure, 2000, Ch. 3, Theorem 3.7.4; Bank et al., 2010; Safey El Din and Schost, 2017, Proposition B.3) and similar arguments as in the proof of Theorem 4.3, we obtain

Corollary 4.7. *There exists a non-empty Zariski open set $\mathcal{U} \subset \mathbb{C}^{n \times n}$ such that for all $A \in \mathcal{U} \cap \mathbb{R}^{n \times n}$, the Zariski closure $\overline{\Gamma_{\mathbb{C}}(f^A) \setminus \text{Crit}_{\mathbb{C}}(f^A)}^{\mathbb{Z}}$ is a one-dimensional algebraic variety in \mathbb{C}^n .*

Proof. Denote $\mathcal{S}_{\mathbb{C}}^{n \times n}$ as the set of symmetric matrices in $\mathbb{C}^{n \times n}$, which can be identified with the space $\mathbb{C}^{\frac{n(n+1)}{2}}$. By similar arguments as in Theorem 4.3, it is easy to see that $\mathbf{0}$ is a regular value of the map

$$\begin{aligned} F : (\mathbb{C}^n \setminus \text{Crit}_{\mathbb{C}}(f)) \times \mathbb{C} \times \mathcal{S}_{\mathbb{C}}^{n \times n} &\rightarrow \mathbb{C}^n \\ (x, \lambda, P) &\mapsto \nabla f(x) - \lambda Px. \end{aligned}$$

Then according to the algebraic version of Thom’s weak transversality theorem (cf. Demazure, 2000, Ch. 3, Theorem 3.7.4; Bank et al., 2010; Safey El Din and Schost, 2017, Proposition B.3), there exists a Zariski closed subset $\Sigma_{\mathbb{C}} \subset \mathcal{S}_{\mathbb{C}}^{n \times n}$ such that for all $P \in \mathcal{S}_{\mathbb{C}}^{n \times n} \setminus \Sigma_{\mathbb{C}}$, $\mathbf{0}$ is a regular value of the map

$$\begin{aligned} F_P : (\mathbb{C}^n \setminus \text{Crit}_{\mathbb{C}}(f)) \times \mathbb{C} &\rightarrow \mathbb{C}^n \\ (x, \lambda) &\rightarrow F(x, \lambda, P). \end{aligned}$$

It follows that $F_P^{-1}(\mathbf{0})$ is either empty or a one-dimensional quasi-affine set of \mathbb{C}^{n+1} . Since $\overline{\Gamma_{\mathbb{C}}(f, P) \setminus \text{Crit}_{\mathbb{C}}(f)}^{\mathbb{Z}}$ is the Zariski closure of the projection of $F_P^{-1}(\mathbf{0})$ on the first n coordinates, we have $\dim(\overline{\Gamma_{\mathbb{C}}(f, P) \setminus \text{Crit}_{\mathbb{C}}(f)}^{\mathbb{Z}}) \leq 1$ for all $P \in \mathcal{S}_{\mathbb{C}}^{n \times n} \setminus \Sigma_{\mathbb{C}}$. As the Zariski closure

$$\Sigma_{\mathbb{C}}^{-1} := \overline{\{P \in \mathcal{S}_{\mathbb{C}}^{n \times n} \mid P^{-1} \in \Sigma_{\mathbb{C}}\}}^{\mathbb{Z}}$$

is an algebraic set in $\mathbb{C}^{\frac{n(n+1)}{2}}$, the set

$$\{A \in \mathbb{C}^{n \times n} \mid AA^T \in \Sigma_{\mathbb{C}}^{-1}\}$$

is an algebraic set in $\mathbb{C}^{n \times n}$. Let $\mathcal{I}_{\mathbb{C}}^{n \times n}$ be the set of invertible matrices in $\mathbb{C}^{n \times n}$. It follows that $\mathcal{U} := \{A \in \mathbb{C}^{n \times n} \mid AA^T \notin \Sigma_{\mathbb{C}}^{-1}\} \cap \mathcal{I}_{\mathbb{C}}^{n \times n}$ is a non-empty Zariski open set of $\mathbb{C}^{n \times n}$. By the definition, for all $A \in \mathcal{U} \cap \mathbb{R}^{n \times n}$, we have $P := (AA^T)^{-1} \in \mathcal{S}_{\mathbb{C}}^{n \times n} \setminus \Sigma_{\mathbb{C}}$ and hence, $\dim(\overline{\Gamma_{\mathbb{C}}(f^A) \setminus \text{Crit}_{\mathbb{C}}(f^A)}^{\mathbb{Z}}) \leq 1$ by Lemma 4.6. \square

4.2. Sufficient criteria for faithful radius

For a given $\mathcal{R} \in \mathbb{R}_+$, we consider the following condition:

Condition 4.8. For any $\mathbf{0} \neq u \in \Gamma_{\mathbb{R}}(f)$ with $\|u\|_2 < \mathcal{R}$, there exist a neighborhood $\mathcal{O}_u \subset \mathbf{B}_{\mathcal{R}}$ of u , a differentiable map $\phi(t) : (a, b) \rightarrow \mathbb{R}^n$ and $\bar{t} \in (a, b)$ such that $\phi((a, b)) = \Gamma_{\mathbb{R}}(f) \cap \mathcal{O}_u$, $\phi(\bar{t}) = u$ and

$$\frac{d(\sum_{i=1}^n \phi_i^2)}{dt}(\bar{t}) \neq 0. \tag{4.3}$$

Theorem 4.9. Suppose that $\mathcal{R} \in \mathbb{R}_+$ satisfies Condition 4.8 and $0 < R < \mathcal{R}$, then $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_R$ is connected. Moreover, if R is an isolation radius, then $\Gamma_{\mathbb{R}}(f) \cap \mathbf{V}_{\mathbb{R}}(f) \cap \mathbf{B}_R = \{\mathbf{0}\}$ and hence R is a faithful radius.

Proof. Suppose that $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_R$ is not connected, then it has a connected component \mathcal{C} such that $\mathbf{0} \notin \mathcal{C}$. Since $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_R$ is closed, \mathcal{C} is closed by Proposition 2.2. Then, the function $\|X\|_2^2$ reaches its minimum on \mathcal{C} at a minimizer $u \in \mathcal{C}$. By the assumption, there exist a neighborhood \mathcal{O}_u of u and a differentiable mapping $\phi(t) : (a, b) \rightarrow \mathbb{R}^n$ and $\bar{t} \in (a, b)$ such that $\phi((a, b)) = \Gamma_{\mathbb{R}}(f) \cap \mathcal{O}_u$ and $\phi(\bar{t}) = u$. By choosing a, b near enough to \bar{t} , we may assume that $\phi((a, b)) \subseteq \mathcal{C} \cap \mathcal{O}_u$. Then, the function $\sum_{i=1}^n \phi_i^2$ reaches its local minimum at \bar{t} , which contradicts (4.3). Hence, $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_R$ is connected.

Now suppose that R is also an isolation radius. Assume to the contrary that there exists $\mathbf{0} \neq v \in \Gamma_{\mathbb{R}}(f) \cap \mathbf{V}_{\mathbb{R}}(f) \cap \mathbf{B}_R$. Since $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_R$ is connected, there exists a path connecting $\mathbf{0}$ and v . Then, f has a local extremum on a relative interior of this path, say u . By the assumption, there exists a differentiable mapping $\phi(t)$ on (a, b) and $\bar{t} \in (a, b)$ as described in the statement. Then the differentiable function $f(\phi(t))$ reaches a local extremum at \bar{t} . By the mean value theorem,

$$0 = \frac{df(\phi)}{dt}(\bar{t}).$$

On the other hand, since R is an isolation radius, $\phi(\bar{t}) \in \Gamma_{\mathbb{R}}(f) \setminus \text{Crit}_{\mathbb{R}}(f)$ and hence there exists $\lambda \neq 0$ such that

$$\frac{\partial f}{\partial x_i}(\phi(\bar{t})) = \lambda \phi_i(\bar{t}), \quad \text{for } i = 1, \dots, n.$$

Therefore,

$$0 = \frac{df(\phi)}{dt}(\bar{t}) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\phi(\bar{t})) \frac{d\phi_i}{dt}(\bar{t}) = \sum_{i=1}^n \lambda \phi_i(\bar{t}) \frac{d\phi_i}{dt}(\bar{t}) = \lambda \frac{d(\sum_{i=1}^n \phi_i^2)}{2dt}(\bar{t}),$$

which contradicts (4.3). Therefore $\Gamma_{\mathbb{R}}(f) \cap \mathbf{V}_{\mathbb{R}}(f) \cap \mathbf{B}_R = \{\mathbf{0}\}$. \square

According to Theorem 4.9, if we can compute $\mathcal{R} \in \mathbb{R}_+$ satisfying Condition 4.8 and an isolation radius R of $\mathbf{0}$ is given, then any $r \in \mathbb{R}_+$ with $r < \min\{R, \mathcal{R}\}$ is a faithful radius of $\mathbf{0}$. We next show how to compute such a \mathcal{R} .

For a given ideal $I \subseteq \mathbb{R}[X]$ with $\dim(I) = 1$, compute its equidimensional decomposition $I = I^{(0)} \cap I^{(1)}$ where $\dim(I^{(i)}) = i$ for $i = 0, 1$. Compute the radical ideal $\sqrt{I^{(1)}} = \langle g_1, \dots, g_s \rangle$ with generators $g_1, \dots, g_s \in \mathbb{R}[X]$. Note that there are efficient algorithms for the equidimensional decomposition of I such that $I^{(0)}$ and $I^{(1)}$ are themselves radical (cf. Aubry et al., 2002, Section 3, and Greuel and Pfister, 2002, Algorithm 4.4.9). Recall that $\|X\|_2^2 = X_1^2 + \dots + X_n^2$. Denote \mathcal{D} as the set of the determinants of the Jacobian matrices $\text{Jac}(g_{i_1}, \dots, g_{i_{n-1}}, \|X\|_2^2)$ for all $\{i_1, \dots, i_{n-1}\} \subset \{1, \dots, s\}$. (Note that $s \geq n - 1$ because $\dim(I^{(1)}) = 1$.) Define

$$\begin{aligned} \mathcal{R}_{I^{(0)}} &:= \min\{r \in \mathbb{R} \setminus \{0\} \mid \exists x \in \mathbf{V}_{\mathbb{C}}(I^{(0)}), \text{ s.t. } x_1^2 + \dots + x_n^2 = r^2\}, \\ \Delta_{I^{(1)}} &:= \{g_1, \dots, g_s\} \cup \mathcal{D}, \\ \mathcal{R}_{I^{(1)}} &:= \inf\{r \in \mathbb{R} \setminus \{0\} \mid \exists x \in \mathbf{V}_{\mathbb{C}}(\Delta_{I^{(1)}}), \text{ s.t. } x_1^2 + \dots + x_n^2 = r^2\}, \\ \mathcal{R}_I &:= \min\{\mathcal{R}_{I^{(0)}}, \mathcal{R}_{I^{(1)}}\}. \end{aligned} \tag{4.4}$$

We have (see also Milnor, 1968, Corollary 2.8):

Lemma 4.10. *If $\dim(I) = 1$, then $\mathcal{R}_I > 0$.*

Proof. Since $I^{(0)}$ is zero-dimensional, we have $\mathcal{R}_{I^{(0)}} > 0$. Now we show that $\mathcal{R}_{I^{(1)}} > 0$. Consider the map

$$\begin{aligned} \Phi : \mathbf{V}_{\mathbb{C}}(I^{(1)}) &\rightarrow \mathbb{C} \\ x &\mapsto x_1^2 + \dots + x_n^2. \end{aligned}$$

Since $\langle g_1, \dots, g_s \rangle$ is radical and equidimensional one, $\mathbf{V}_{\mathbb{C}}(\Delta_{I^{(1)}})$ consists of the singular locus of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ and the set of critical points of Φ . Since $\dim(\mathbf{V}_{\mathbb{C}}(I^{(1)})) = 1$, its singular locus is zero-dimensional. By the algebraic Sard’s theorem (Shafarevich, 1977), there are only finitely many critical values of the map Φ (note that if Φ is not dominant, the conclusion is clearly true). Hence, the set $\Phi(\mathbf{V}_{\mathbb{C}}(\Delta_{I^{(1)}}))$ is finite and $\mathcal{R}_{I^{(1)}} > 0$. \square

Theorem 4.11. *Suppose that an ideal $I \subseteq \mathbb{R}[X]$ and a $\mathcal{R} \in \mathbb{R}_+$ satisfy: (i) $\dim(I) = 1$; (ii) $\mathcal{R} < \mathcal{R}_I$; (iii) $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_{\mathcal{R}} = \mathbf{V}_{\mathbb{R}}(I) \cap \mathbf{B}_{\mathcal{R}}$, then Condition 4.8 holds for \mathcal{R} .*

Proof. Fix a $\mathbf{0} \neq u \in \Gamma_{\mathbb{R}}(f)$ with $\|u\|_2 < \mathcal{R}$. Since $\mathcal{R} < \mathcal{R}_I$ and $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_{\mathcal{R}} = \mathbf{V}_{\mathbb{R}}(I) \cap \mathbf{B}_{\mathcal{R}}$, by Corollary 3.4 and the definition of \mathcal{R}_I , we have $\mathbf{V}_{\mathbb{R}}(I) \cap \mathbf{B}_{\mathcal{R}} = \mathbf{V}_{\mathbb{R}}(I^{(1)}) \cap \mathbf{B}_{\mathcal{R}}$ and hence $u \in \mathbf{V}_{\mathbb{R}}(I^{(1)})$. Because $u \notin \mathbf{V}_{\mathbb{R}}(\Delta_{I^{(1)}})$, there is a Jacobian matrix of the form

$$M = \begin{bmatrix} \frac{\partial g_{i_1}}{\partial x_1}(u) & \dots & \frac{\partial g_{i_1}}{\partial x_n}(u) \\ \vdots & \vdots & \vdots \\ \frac{\partial g_{i_{n-1}}}{\partial x_1}(u) & \dots & \frac{\partial g_{i_{n-1}}}{\partial x_n}(u) \\ u_1 & \dots & u_n \end{bmatrix} \tag{4.5}$$

with full rank.

Let $\tilde{g}(X) = \sum_{i=1}^n u_i X_i - \sum_{i=1}^n u_i^2$ and $\tilde{I}^{(1)} = \langle g_{i_1}, \dots, g_{i_{n-1}} \rangle$. Define a function $Y = G(X) := (g_{i_1}(X), \dots, g_{i_{n-1}}(X), \tilde{g}(X))$, then we have $G(u) = \mathbf{0}$ and the Jacobian of $G(X)$ at u is nonsingular. Hence, by the inverse function theorem, $G(X)$ is an invertible function in a neighborhood \mathcal{O}_u of u . Without loss of generality, we can assume that $\mathcal{O}_u \subseteq \mathbf{B}_{\mathcal{R}}$. Thus, an invertible function $X = G^{-1}(Y) = (G_1^{-1}(Y), \dots, G_n^{-1}(Y))$ exists in some neighborhood $\mathcal{O}_{\mathbf{0}}$ of $\mathbf{0}$. Moreover, $X = G^{-1}(Y)$ is differentiable in $\mathcal{O}_{\mathbf{0}}$. Define $\phi(t) = (\phi_i(t))$ with $\phi_i(t) = G_i^{-1}(\mathbf{0}, \dots, \mathbf{0}, t)$. Then, there is an interval (a, b) such that $\phi((a, b)) = \mathbf{V}_{\mathbb{R}}(\tilde{I}^{(1)}) \cap \mathcal{O}_u$. Moreover, we have $\mathbf{0} \in (a, b)$ and $\phi(\mathbf{0}) = u$. Then we have

$$M \cdot \begin{bmatrix} \frac{\partial G_1^{-1}}{\partial Y_n}(\mathbf{0}) \\ \vdots \\ \frac{\partial G_{n-1}^{-1}}{\partial Y_n}(\mathbf{0}) \\ \frac{\partial G_n^{-1}}{\partial Y_n}(\mathbf{0}) \end{bmatrix} = \begin{bmatrix} \frac{\partial Y_1}{\partial Y_n}(\mathbf{0}) \\ \vdots \\ \frac{\partial Y_{n-1}}{\partial Y_n}(\mathbf{0}) \\ \frac{\partial \sum_{i=1}^n (G_i^{-1})^2}{2\partial Y_n}(\mathbf{0}) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{d \sum_{i=1}^n \phi_i^2}{2dt}(\mathbf{0}) \end{bmatrix}$$

By the implicit function theorem, there is i_0 such that $\frac{\partial G_{i_0}^{-1}}{\partial Y_n}(\mathbf{0}) \neq 0$. Since the matrix M in (4.5) is nonsingular, it implies $\frac{d \sum_{i=1}^n \phi_i^2}{dt}(\mathbf{0}) \neq 0$. Recalling Condition 4.8, since $\Gamma_{\mathbb{R}}(f) \cap \mathcal{O}_u = \mathbf{V}_{\mathbb{R}}(I^{(1)}) \cap \mathcal{O}_u$, it remains to prove that $\mathbf{V}_{\mathbb{R}}(I^{(1)}) \cap \mathcal{O}_u = \mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)}) \cap \mathcal{O}_u$.

It suffices to prove that $\mathbf{V}_{\mathbb{C}}(I^{(1)}) \cap \mathcal{O} = \mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)}) \cap \mathcal{O}$ for some Zariski open set $\mathcal{O} \subseteq \mathbb{C}^n$ containing u . Let $\mathbf{V}_{\mathbb{C}}(I^{(1)}) = V_1 \cup \dots \cup V_s$ and $\mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)}) = \tilde{V}_1 \cup \dots \cup \tilde{V}_t$ be the irreducible decompositions of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ and $\mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)})$, respectively. Since the first $n - 1$ rows of M is linear independent, there is a unique irreducible component, say \tilde{V}_1 , of $\mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)})$ containing u and \tilde{V}_1 is smooth of dimension one at u . Let V_1 be an irreducible component of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ containing u . Since $I^{(1)} \supseteq \tilde{I}^{(1)}$,

we have $\mathbf{V}_{\mathbb{C}}(I^{(1)}) \subseteq \mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)})$ and hence $V_1 \subseteq \tilde{V}_1$. Because $\dim(V_1) = 1$, it follows that $V_1 = \tilde{V}_1$ which also implies that V_1 is the unique irreducible component of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ containing u . Let $\mathcal{O} = \mathbb{C}^n \setminus (\bigcup_{i=2}^s V_i \cup \bigcup_{i=2}^t \tilde{V}_i)$, then we have $\mathbf{V}_{\mathbb{C}}(I^{(1)}) \cap \mathcal{O} = \mathbf{V}_{\mathbb{C}}(\tilde{I}^{(1)}) \cap \mathcal{O}$ which ends the proof. \square

Combining Theorems 4.9 and 4.11, we obtain

Theorem 4.12. *Suppose that an ideal $I \subseteq \mathbb{R}[X]$ and a $\mathcal{R} \in \mathbb{R}_+$ satisfy: (i) $\dim(I) = 1$; (ii) $\mathcal{R} < \mathcal{R}_I$; (iii) $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_{\mathcal{R}} = \mathbf{V}_{\mathbb{R}}(I) \cap \mathbf{B}_{\mathcal{R}}$, then any isolation radius $R \in \mathbb{R}_+$ of $\mathbf{0}$ with $R < \mathcal{R}$ is a faithful radius of $\mathbf{0}$.*

Let $\boldsymbol{\gamma} = \{\gamma_{i,j} \mid i, j = 1, \dots, n\}$ where $\gamma_{i,j} := \frac{\partial f}{\partial X_i} X_j - \frac{\partial f}{\partial X_j} X_i$, then $\Gamma_{\mathbb{R}}(f) = \mathbf{V}_{\mathbb{R}}(\langle \boldsymbol{\gamma} \rangle)$. If $\dim(\langle \boldsymbol{\gamma} \rangle) = 1$, let $\mathcal{R}_{\langle \boldsymbol{\gamma} \rangle}$ be defined as in (4.4).

Corollary 4.13. *If $\dim(\langle \boldsymbol{\gamma} \rangle) = 1$, then any isolation radius R with $R < \mathcal{R}_{\langle \boldsymbol{\gamma} \rangle}$ is a faithful radius of $\mathbf{0}$.*

Note that the ideal $\langle \boldsymbol{\gamma} \rangle$ may not be one-dimensional. Let $\mathcal{G} := \mathbf{I}(\overline{\Gamma_{\mathbb{C}}(f) \setminus \text{Crit}_{\mathbb{C}}(f)}^{\mathcal{Z}})$, i.e., the vanishing ideal of $\overline{\Gamma_{\mathbb{C}}(f) \setminus \text{Crit}_{\mathbb{C}}(f)}^{\mathcal{Z}}$ in $\mathbb{R}[X]$. According to Corollary 4.7, we have $\dim(\mathcal{G}) = 1$ up to a generic linear change of coordinates. Moreover,

Proposition 4.14. *The equality $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_R = \mathbf{V}_{\mathbb{R}}(\mathcal{G}) \cap \mathbf{B}_R$ holds for any isolation radius R .*

Proof. Let $V_1 \cup \dots \cup V_s \cup V_{s+1} \cup \dots \cup V_t$ be the decomposition of $\Gamma_{\mathbb{C}}(f)$ as a union of irreducible components. Assume that $V_i \not\subseteq \text{Crit}_{\mathbb{C}}(f)$ for $1 \leq i \leq s$ and $V_j \subseteq \text{Crit}_{\mathbb{C}}(f)$ for $s+1 \leq j \leq t$. Let $V^{(1)} = V_1 \cup \dots \cup V_s$ and $V^{(2)} = V_{s+1} \cup \dots \cup V_t$, then $\mathbf{V}_{\mathbb{C}}(\mathcal{G}) = V^{(1)}$ and $V^{(2)} \cap \mathbf{B}_R \subseteq \{\mathbf{0}\}$. We have $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_R = \Gamma_{\mathbb{C}}(f) \cap \mathbf{B}_R = (V^{(1)} \cap \mathbf{B}_R) \cup (V^{(2)} \cap \mathbf{B}_R) \subseteq (\mathbf{V}_{\mathbb{R}}(\mathcal{G}) \cap \mathbf{B}_R) \cup \{\mathbf{0}\}$. Since $\Gamma_{\mathbb{R}}(f) \setminus \text{Crit}_{\mathbb{R}}(f) \subset \mathbf{V}_{\mathbb{R}}(\mathcal{G})$, by Corollary 3.4, $\mathbf{0} \in \mathbf{V}_{\mathbb{R}}(\mathcal{G})$ and hence $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_R \subseteq \mathbf{V}_{\mathbb{R}}(\mathcal{G}) \cap \mathbf{B}_R$. It is clear that $\Gamma_{\mathbb{R}}(f) \cap \mathbf{B}_R \supseteq \mathbf{V}_{\mathbb{R}}(\mathcal{G}) \cap \mathbf{B}_R$ and thus the conclusion follows. \square

Corollary 4.15. *Suppose that $\dim(\mathcal{G}) = 1$ and R is an isolation radius of f . Then, any $r \in \mathbb{R}_+$ with $r < \min\{R, \mathcal{R}_{\mathcal{G}}\}$ is a faithful radius of $\mathbf{0}$.*

Provided that an isolation radius of $\mathbf{0}$ is known, we now present an algorithm to compute faithful radius of $\mathbf{0}$.

Algorithm 4.16. FaithfulRadius(f, R_{iso})

Input: A polynomial $f \in \mathbb{R}[X]$ with $\mathbf{0}$ as an isolated real critical point and an isolation radius R_{iso} of $\mathbf{0}$.

Output: $R \in \mathbb{R}_+$ such that any $0 < r < R$ is a faithful radius of $\mathbf{0}$.

1. If $\dim(\langle \boldsymbol{\gamma} \rangle) = 1$, then let $I = \langle \boldsymbol{\gamma} \rangle$; otherwise, make a linear change of coordinates of f such that $\dim(\mathcal{G}) = 1$ and let $I = \mathcal{G}$;
2. Compute the equidimensional decomposition $I = I^{(0)} \cap I^{(1)}$ and the set $\Delta_{I^{(1)}}$ as defined in (4.4);
3. Compute elimination ideals $I_n^{(0)} := (I^{(0)} + (\|X\|_2^2 - X_{n+1})) \cap \mathbb{R}[X_{n+1}]$ and $I_n^{(1)} := ((\Delta_{I^{(1)}}) + (\|X\|_2^2 - X_{n+1})) \cap \mathbb{R}[X_{n+1}]$;
4. Compute the isolation intervals $\{[a_i, b_i] \mid i = 1, \dots, t\}$ of $\mathbf{V}_{\mathbb{R}}(I_n^{(0)} \cdot I_n^{(1)})$;
5. Let $\mathcal{R} = \min\{\sqrt{a_i} \mid a_i > 0, i = 1, \dots, t\}$ and return $R = \min\{\mathcal{R}, R_{\text{iso}}\}$.

Theorem 4.17. *Algorithm 4.16 runs successfully and is correct.*

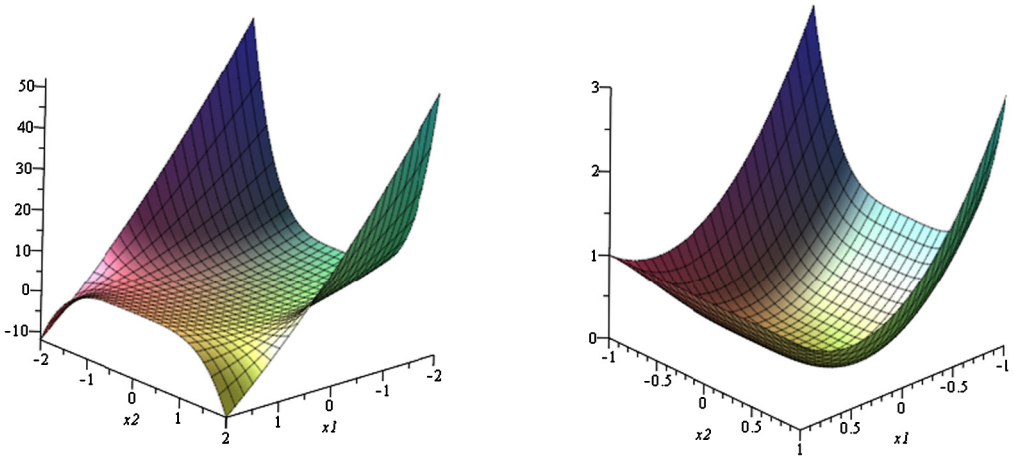


Fig. 1. The graphs of f in Example 4.18.

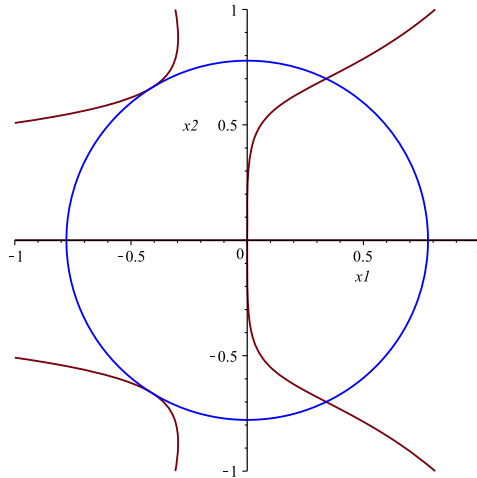


Fig. 2. The curve of tangency of f in Example 4.18. (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

Proof. According to the proof of Proposition 4.14, we have $\mathbf{0} \in \mathbf{V}_{\mathbb{R}}(I^{(1)})$ and hence $0 \in \mathbf{V}_{\mathbb{R}}(I_n^{(0)} \cdot I_n^{(1)})$ by the definition of $\Delta_{I^{(1)}}$. Then, we have $\mathcal{R} < \mathcal{R}_1$ in Step 5 since $[a_i, b_i]$'s are isolation intervals of $\mathbf{V}_{\mathbb{R}}(I_n^{(0)} \cdot I_n^{(1)})$. Then, by Corollary 4.7 and the proof of Lemma 4.10, the algorithm runs successfully. Its correctness can be seen by combining Theorem 4.12 and Corollary 4.15. \square

Example 4.18. Consider the polynomial $f = X_1^2 + (1 - X_1)X_2^4$ discussed in the introduction. The origin $\mathbf{0}$ is an isolated real critical point and degenerate. The graphs of f are shown in Fig. 1. On the left hand side, the graph is drawn with the variables X_1, X_2 varying in the range $[-2, 2]$. It seems from this graph that $\mathbf{0}$ is a saddle point. However, if we zoom in, then we get the graph on the right hand side which indicates that $\mathbf{0}$ is in fact a strict local minimizer. Now we use Algorithm 4.16 to obtain a faithful radius of $\mathbf{0}$.

It is easy to check that $R_{\text{iso}} = 1$ is an isolation radius of $\mathbf{0}$ and $\gamma = \{4X_1^2X_2^3 - X_2^5 - 4X_1X_2^3 + 2X_1X_2\}$. The curve of tangency $\Gamma_{\mathbb{R}}(f) = \mathbf{V}_{\mathbb{R}}(\langle \gamma \rangle)$ is shown (red) in Fig. 2. Since $\dim(\langle \gamma \rangle) = 1$, we let $I = \langle \gamma \rangle$.

We implement Algorithm 4.16 in the software MAPLE. With inputs f and R_{iso} , we get the return $R = \frac{\sqrt{767451466998008631606300139861}}{1125899906842624} \approx 0.778 < 1$. The circle centered at $\mathbf{0}$ with radius R is shown (blue) in Fig. 2. Hence, any $r < R$ is a faithful radius of $\mathbf{0}$.

4.3. On the computation of isolation radius

As we have seen, if an isolation radius of $\mathbf{0}$ is available, a faithful radius of $\mathbf{0}$ can be obtained by Algorithm 4.16. To end this section, we propose some strategies to compute isolation radius of $\mathbf{0}$.

Let $\mathcal{C} = (\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n})$. If $\dim(\mathcal{C}) = 0$, then an isolation radius of $\mathbf{0}$ can be computed by algorithms in, for example, Giusti et al. (1995, 2001), Faugère et al. (1993), Alonso et al. (1996), Rouillier (1999) for zero-dimensional polynomial systems.

Assume that $\dim(\mathcal{C}) > 0$. We now borrow the idea from Aubry et al. (2002) which aims to compute one point on each semi-algebraically connected component of a real algebraic variety by the critical point method of a distance function. Compute the equidimensional decomposition $\mathcal{C} = \mathcal{C}^{(0)} \cap \mathcal{C}^{(1)} \cap \dots \cap \mathcal{C}^{(t)}$ where $\mathcal{C}^{(k)}$ is radical and of dimension k for each $k = 0, \dots, t$. For an efficient algorithm of such decomposition, see Aubry et al. (2002, Section 3). Then, we can compute the minimal distance $d^{(k)}$ of $\mathbf{V}_{\mathbb{R}}(\mathcal{C}^{(k)}) \setminus \{\mathbf{0}\}$ to $\mathbf{0}$ for each k and choose a positive number less than the smallest one as an isolation radius. Suppose that $\mathcal{C}^{(k)} = \langle h_1, \dots, h_l \rangle$. Let $\mathcal{M}(\mathcal{C}^{(k)})$ be the set of h_1, \dots, h_l and all the $(n - k + 1, n - k + 1)$ minors of the Jacobian matrix $\text{Jac}(h_{i_1}, \dots, h_{i_{n-k}}, \|X\|_2^2)$ for all $\{i_1, \dots, i_{n-k}\} \subset \{1, \dots, l\}$. Consider the map

$$\begin{aligned} \Phi : \mathbf{V}_{\mathbb{C}}(\mathcal{C}^{(k)}) &\rightarrow \mathbb{C} \\ x &\mapsto x_1^2 + \dots + x_n^2. \end{aligned}$$

Then, $\mathbf{V}_{\mathbb{C}}(\mathcal{M}(\mathcal{C}^{(k)}))$ consists of the singular locus $\text{Sing}(\mathcal{C}^{(k)})$ of $\mathbf{V}_{\mathbb{C}}(\mathcal{C}^{(k)})$ and the set of critical points of Φ . By the first part in the proof of Aubry et al. (2002, Theorem 2.3), the point of $\mathbf{V}_{\mathbb{R}}(\mathcal{C}^{(k)}) \setminus \{\mathbf{0}\}$ at the minimal distance to $\mathbf{0}$ is contained in $\mathbf{V}_{\mathbb{R}}(\mathcal{M}(\mathcal{C}^{(k)}))$. Compute the elimination ideal $\mathcal{M}_n(\mathcal{C}^{(k)}) = \langle \mathcal{M}(\mathcal{C}^{(k)}), \|X\|_2^2 - X_{n+1} \rangle \cap \mathbb{R}[X_{n+1}]$ and then we have $(d^{(k)})^2 \in \mathbf{V}_{\mathbb{R}}(\mathcal{M}_n(\mathcal{C}^{(k)}))$. If $\mathcal{M}_n(\mathcal{C}^{(k)}) \neq \langle 0 \rangle$, then the smallest positive real root in $\mathbf{V}_{\mathbb{R}}(\mathcal{M}_n(\mathcal{C}^{(k)}))$, which can be obtained by any real root isolation algorithm for univariate polynomials, is a lower bound of $(d^{(k)})^2$. If $\mathcal{M}_n(\mathcal{C}^{(k)}) = \langle 0 \rangle$, by Sard's theorem, it happens if and only if the set $\Phi(\text{Sing}(\mathcal{C}^{(k)}))$ is infinite. In this case, we can replace \mathcal{C} by $\mathcal{M}(\mathcal{C}^{(k)})$ and repeat the above procedure recursively. Since $\dim(\text{Sing}(\mathcal{C}^{(k)})) < k$, this process will finitely terminate and return an isolation radius of $\mathbf{0}$.

Alternatively, when $\dim(\mathcal{C}) > 0$, we can compute isolation radius of $\mathbf{0}$ by testing the emptiness of a real algebraic variety. Adding two new variables X_{n+1} and X_{n+2} , a $R \in \mathbb{R}_+$ is an isolation radius of $\mathbf{0}$ if and only if the following polynomial system has no real root

$$\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}, \|X\|_2^2 + X_{n+1}^2 - R^2, \|X\|_2^2 \cdot X_{n+2} - 1.$$

Hence, we can set an initial R and test the emptiness of the real algebraic variety generated by the above polynomials. If it is empty, then R is an isolation radius; otherwise, try $R/2$ and repeat. For algorithms of such tests, see Aubry et al. (2002, Section 4) and Safey El Din and Schost (2003).

5. Certificates of types of degenerate critical points

If $R \in \mathbb{R}_+$ is a faithful radius of the isolated real critical point $\mathbf{0}$ of f , then we can compute the extrema f_R^{\min} and f_R^{\max} in (1.2) to classify the type of $\mathbf{0}$ by Theorem 3.7. To deal with the issues when computing f_R^{\min} and f_R^{\max} as mentioned in the introduction, we next show that how to decide the type of $\mathbf{0}$ by means of real root isolation of zero-dimensional polynomial systems. Recall the notation S_r in (1.1).

Proposition 5.1. *Suppose that $\mathcal{R} \in \mathbb{R}_+$ satisfies Condition 4.8 and $0 < R < \mathcal{R}$. Then for any $\mathbf{0} \neq u \in \Gamma_{\mathbb{R}}(f) \cap \mathbb{B}_R$, there exists a continuous map $\varphi(t) : [a, b] \rightarrow \Gamma_{\mathbb{R}}(f) \cap \mathbb{B}_R$ with $\varphi(a) = u$, $\varphi(b) \in \mathbb{S}_R$ and $\mathbf{0} \notin \varphi([a, b])$.*

Proof. Consider the following semi-algebraic set

$$S := \Gamma_{\mathbb{R}}(f) \cap \{x \in \mathbb{R}^n \mid \|u\|_2^2/2 \leq \|x\|_2^2 \leq R^2\}.$$

Let \mathcal{C} be the connected component of S containing u . If $\mathcal{C} \cap \mathbb{S}_R \neq \emptyset$, then the conclusion follows since \mathcal{C} is path connected. Otherwise, the function $\|X\|_2^2$ reaches its maximum on \mathcal{C} at a maximizer in \mathcal{C} . Then we can get a contradiction using arguments similar to the first part of the proof of Theorem 4.9. \square

For any $r \in \mathbb{R}_+$, comparing with Corollary 3.3, define

$$f_r^- := \min\{f(x) \mid x \in \Gamma_{\mathbb{R}}(f) \cap \mathbb{S}_r\} \quad \text{and} \quad f_r^+ := \max\{f(x) \mid x \in \Gamma_{\mathbb{R}}(f) \cap \mathbb{S}_r\}. \tag{5.1}$$

Theorem 5.2. *Suppose that $\mathcal{R} \in \mathbb{R}_+$ satisfies Condition 4.8. Then for any isolation radius $R < \mathcal{R}$, it holds that*

- (i) *the origin $\mathbf{0}$ is a local minimizer if and only if $f_R^- > 0$;*
- (ii) *the origin $\mathbf{0}$ is a local maximizer if and only if $f_R^+ < 0$;*
- (iii) *the origin $\mathbf{0}$ is a saddle point if and only if $f_R^+ > 0 > f_R^-$.*

Proof. By Theorem 4.9, R is a faithful radius of $\mathbf{0}$. According to Theorem 3.7 and Definition 3.6 (ii), the “only if” parts in (i), (ii) and the “if” part in (iii) are clear.

(i). “if” part. Suppose that $f_R^- > 0$, then we have $f_R^{\min} = 0$. Otherwise, by Corollary 3.3, there exists $\mathbf{0} \neq u \in \Gamma_{\mathbb{R}}(f) \cap \mathbb{B}_R \setminus \mathbb{S}_R$ such that $f(u) < 0$. By Proposition 5.1, there exists a continuous map $\varphi(t) : [a, b] \rightarrow \Gamma_{\mathbb{R}}(f) \cap \mathbb{B}_R$ with $\varphi(a) = u$ and $\varphi(b) \in \mathbb{S}_R$. Then we have a continuous function $g(t) := f(\varphi(t)) : [a, b] \rightarrow \mathbb{R}$ such that $g(a) < 0$ and $g(b) > 0$. By the mean value theorem, there exists $\bar{t} \in (a, b)$ such that $g(\bar{t}) = f(\varphi(\bar{t})) = 0$. Since that $\mathbf{0} \neq \varphi(\bar{t}) \in \Gamma_{\mathbb{R}}(f) \cap \mathbb{B}_R$ by Proposition 5.1 and R is a faithful radius, we get a contradiction;

Similarly, we can prove (ii) and then (iii) follows. \square

For any $r \in \mathbb{R}_+$, let $\mathbb{S}_{r, \mathbb{C}} = \{x \in \mathbb{C}^n \mid \sum_{i=1}^n x_i^2 = r^2\}$. Recall the definition \mathcal{R}_I for an ideal I in (4.4).

Proposition 5.3. *Given an ideal $I \subseteq \mathbb{R}[X]$ with $\dim(I) = 1$, the system $\mathbf{V}_{\mathbb{C}}(I) \cap \mathbb{S}_{R, \mathbb{C}}$ is zero-dimensional for any $0 < R < \mathcal{R}_I$.*

Proof. It only needs to prove that $\mathbf{V}_{\mathbb{C}}(I^{(1)}) \cap \mathbb{S}_{R, \mathbb{C}}$ is zero-dimensional. Let $V_1 \cup \dots \cup V_s$ be the decomposition of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ as a union of irreducible components. Fix an $1 \leq i \leq s$. If $V_i \cap \mathbb{S}_{R, \mathbb{C}} \neq \emptyset$, then we show that $V_i \not\subseteq \mathbb{S}_{R, \mathbb{C}}$. To the contrary, assume that $V_i \subseteq \mathbb{S}_{R, \mathbb{C}}$. By the definition of \mathcal{R}_I and the proof of Lemma 4.10, we have $V_i \cap V_j = \emptyset$ for each $j \neq i$ since $V_i \cap V_j$ is contained in the singular locus of $\mathbf{V}_{\mathbb{C}}(I^{(1)})$. Then for any point p in the nonsingular part of V_i , it holds that $T_p(V_i) = T_p(\mathbf{V}_{\mathbb{C}}(I^{(1)}))$ where $T_p(V_i)$ and $T_p(\mathbf{V}_{\mathbb{C}}(I^{(1)}))$ denote the tangent spaces of V_i and $\mathbf{V}_{\mathbb{C}}(I^{(1)})$ at p , respectively. Then, for any $h \in \mathbf{I}(V_i)$, the differential of h at p can be expressed as a linear combination of the differentials of g_1, \dots, g_s (the generators of $\sqrt{I^{(1)}}$) at p . In particular, by the assumption that $V_i \subseteq \mathbb{S}_{R, \mathbb{C}}$, it holds for $h := \|X\|_2^2 - R^2 \in \mathbf{I}(V_i)$. Hence, all the determinants in the set \mathcal{D} in (4.4) vanish at any $p \in V_i$ since $\dim(I^{(1)}) = 1$. Consequently, we have $p \in \mathbf{V}_{\mathbb{C}}(\Delta_{I^{(1)}})$. By the definition, it implies that $R \geq \mathcal{R}_I$ which is a contradiction. Therefore, if $V_i \cap \mathbb{S}_{R, \mathbb{C}} \neq \emptyset$, then $V_i \not\subseteq \mathbb{S}_{R, \mathbb{C}}$ and $\dim(V_i \cap \mathbb{S}_{R, \mathbb{C}}) = \dim(V_i) - 1 = 0$ by Krull’s Principal Ideal Theorem (Kunz, 1988, Chap. V, Corollary 3.2). The conclusion follows. \square

Recall the definition of \mathcal{Y} and Algorithm 4.16. We now give an algorithm to decide the type of the isolated real critical point $\mathbf{0}$ of f .

Algorithm 5.4. Type(f, R_{iso})

Input: A polynomial $f \in \mathbb{R}[X]$ with $\mathbf{0}$ as an isolated real critical point and an isolation radius R_{iso} of $\mathbf{0}$.

Output: The type of $\mathbf{0}$ as a critical point of f .

1. If $\dim(\langle \mathbf{y} \rangle) = 1$, then let $I = \langle \mathbf{y} \rangle$; otherwise, make a linear change of coordinates of f such that $\dim(\mathcal{G}) = 1$ and let $I = \mathcal{G}$;
2. Let $R = \text{FaithfulRadius}(f, R_{\text{iso}})$ and fix a radius $0 < r < R$;
3. Let $\bar{I} = I + (\|X\|_2^2 - r^2, f - X_{n+1}) \subseteq \mathbb{R}[X, X_{n+1}]$;
4. Compute intervals $\{[a_i, b_i] \mid i = 1, \dots, s\}$ such that $0 \notin [a_i, b_i]$ for each i and the coordinate X_{n+1} of every point in $\mathbf{V}_{\mathbb{R}}(\bar{I}) \subseteq \mathbb{R}^{n+1}$ lies in some unique $[a_i, b_i]$;
5. Let $m = \min\{a_i, i = 1, \dots, s\}$ and $M = \max\{b_i, i = 1, \dots, s\}$;
6. If $m > 0$, return “local minimizer”; if $M < 0$, return “local maximizer”; if $m < 0 < M$, return “saddle point”.

Theorem 5.5. Algorithm 5.4 runs successfully and is correct.

Proof. By Proposition 4.14 and Algorithm 4.16, f_r^- and f_r^+ respectively equal the minimal and maximal coordinates X_{n+1} of the points in $\mathbf{V}_{\mathbb{R}}(\bar{I})$. Since \bar{I} is zero-dimensional by Proposition 5.3 and f_r^-, f_r^+ are nonzero by Theorem 5.2, the isolation intervals $[a_i, b_i]$'s in step 5 can be obtained. Again, by Theorem 5.2, the outputs of Algorithm is correct. \square

Example 4.18 continued. We have shown that any $0 < r < R \approx 0.778 < 1$ is a faithful radius of $\mathbf{0}$. We set $r = \frac{7}{18}$ in Step 2 of Algorithm 5.4. In Step 4, by the command `Isolate` in MAPLE which uses the Rational Univariate Representations (RUR) method (Rouillier, 1999) for zero-dimensional system, we obtain that $m = \frac{76810939241945}{562949953421312}$ and $M = \frac{437849963772149}{1125899906842624}$ in step 5. Therefore, we can claim that $\mathbf{0}$ is a local minimizer of f by Step 6 of Algorithm 5.4. Our algorithm is implemented with MAPLE 18 on a laptop with two 1.3 GHz cores and 8GB RAM. The total CPU time used for solving Example 4.18 is less than one second.

Example 5.6. Consider the following polynomial (cf. Nie, 2015; Nie and Ranestad, 2009)

$$f(X_1, X_2, X_3) = 47X_1^5 + 5X_1X_2^4 + 33X_3^5 - 95X_1^4 - 47X_1X_3^3 + 51X_2^2X_3^2 - 92X_1X_3^2 - 70X_2^2X_3 + 21X_2^2.$$

It can be checked that $\mathbf{0}$ is a degenerate critical point and moreover the set $\text{Crit}_{\mathbb{C}}(f)$ is zero-dimensional. Hence, $\mathbf{0}$ is isolated real critical point. Using the command `Isolate` in MAPLE, we get an isolation radius $R_{\text{iso}} = \frac{70375577207295}{140737488355328} \approx 0.50$. Running Algorithm 4.16 with inputs f and R_{iso} , we get the output $R = \frac{459690419250099}{4503599627370496} \approx 0.102$. Setting $r = \frac{2}{39}$ in Step 2 of Algorithm 5.4, we obtain $m = -\frac{428092208351331}{2251799813685248}$ and $M = \frac{13589527442797}{70368744177664}$. Thus, $\mathbf{0}$ is a saddle point of f . In fact, it can be certified by letting $X_1 = X_2 = 0$ in f . The total CPU time used for solving this example is less than five seconds.

Example 5.7. Consider the polynomial $f(X_1, X_2, X_3) = X_1^2 + X_2^4 + X_3^4 - 4X_1X_2X_3$ with $\mathbf{0}$ as a degenerate critical point. It is shown in Cushing (1975) that the method proposed therein fails to test the type of $\mathbf{0}$. For any $\varepsilon > 0$, we have $f(\varepsilon^2, \varepsilon, \varepsilon) < 0$ and $f(-\varepsilon, \varepsilon, \varepsilon) > 0$. Letting $\varepsilon \rightarrow 0$, we get that $\mathbf{0}$ is a saddle point of f . Since the set $\text{Crit}_{\mathbb{C}}(f) = \{\mathbf{0}\}$, we set an isolation radius $R_{\text{iso}} = 1$. Running Algorithm 4.16 with inputs f and R_{iso} , we get the output $R = \frac{1}{2}$. Setting $r = \frac{1}{4}$ in Step 2 of Algorithm 5.4, we obtain $m = -\frac{90700979328567}{9007199254740992}$ and $M = \frac{5629499534213}{35184372088832}$. Thus, we can detect that $\mathbf{0}$ is a saddle point of f . The total CPU time used for solving this example is less than one second.

Remark 5.8. Clearly, determining the type of the critical point $\mathbf{0}$ of the polynomial f is equivalent to checking the truth of the following first-order sentences

$$\begin{aligned} & \forall r \exists x_1 \cdots \exists x_n (r = 0) \vee ((x_1^2 + \cdots + x_n^2 \leq r^2) \wedge (f(x_1, \dots, x_n) > 0)), \\ & \forall r \exists x_1 \cdots \exists x_n (r = 0) \vee ((x_1^2 + \cdots + x_n^2 \leq r^2) \wedge (f(x_1, \dots, x_n) < 0)), \end{aligned}$$

where \vee and \wedge respectively denote the logical connectives “or” and “and”, which can be done by *quantifier elimination* algorithms (cf. Basu et al., 2006; Caviness and Johnson, 1998). We now use the `QuantifierElimination` command (cf. Chen and Moreno Maza, 2014a, 2014b, 2014c, 2016) of the `RegularChains`¹ library in `MAPLE` to determine the type of $\mathbf{0}$ in Example 4.18, 5.6 and 5.7 treated as quantifier elimination problems. As a result, Example 4.18 was solved immediately. However, for Example 5.6 and 5.7, the `QuantifierElimination` command kept running for several hours without any output, which shows the efficiency of our method.

To conclude this section, we would like to point out that the cost of running Algorithms 4.16 and 5.4 can be reduced if some factor decomposition of f is available. The following propositions show that the problem of classifying the isolated real critical point $\mathbf{0}$ of f reduces to the case when f is square-free and each of its factors vanishes at $\mathbf{0}$.

Proposition 5.9. Suppose $f(X) = g(X)h(X)^2$ where $g(X), h(X) \in \mathbb{R}[X]$. Then,

- Case 1. $g(\mathbf{0}) \neq 0$. If $g_0 > 0$, then $\mathbf{0}$ is a local minimizer of $f(X)$; otherwise, $\mathbf{0}$ is a local maximizer. Here, g_0 denotes the constant term of $g(X)$.
- Case 2. $g(\mathbf{0}) = 0$. If $\nabla g(\mathbf{0}) \neq \mathbf{0}$, then $\mathbf{0}$ is a saddle point of $f(X)$; otherwise, $\mathbf{0}$ is of the same type as a common critical point of $f(X)$ and $g(X)$.

Proof. If $g(\mathbf{0}) \neq 0$ then there exists an open neighborhood $U \subset \mathbb{R}^n$ of $\mathbf{0}$ such that either $g > 0$ or $g < 0$ on U . This implies easily that $\mathbf{0}$ is a local minimizer or maximizer of f .

Assume that $g(\mathbf{0}) = 0$ and $\nabla g(\mathbf{0}) \neq \mathbf{0}$. We have that $\mathbf{0}$ is not a local extremal point of g . Consequently, for any open neighborhood $U \subset \mathbb{R}^n$ of $\mathbf{0}$, there exist points $u, v \in U$ such that $g(u) < 0 < g(v)$. On the other hand, the algebraic set $h^{-1}(0)$ has dimension $< n$. By continuity of g , we may assume that $u, v \notin h^{-1}(0)$. Therefore, $f(u) < 0 < f(v)$, and so $\mathbf{0}$ is a saddle point of f .

Finally, suppose that $g(\mathbf{0}) = 0$ and $\nabla g(\mathbf{0}) = \mathbf{0}$. Because for any $x \in \mathbb{R}^n$ with $f(x) \neq 0$, $f(x)$ and $g(x)$ have the same sign, $\mathbf{0}$ is of the same type as a common critical point of f and g . \square

Proposition 5.10. Suppose that $\mathbf{0}$ is a critical point of $f \in \mathbb{R}[X]$ and $f = f_1 \cdot f_2$ where $f_1, f_2 \in \mathbb{R}[X]$. If $f_1(\mathbf{0}) \neq 0$, then

- (i) the origin $\mathbf{0}$ is a saddle point of f if and only if $\mathbf{0}$ is a saddle point of f_2 ;
- (ii) if $f_1(\mathbf{0}) > 0$ ($f_1(\mathbf{0}) < 0$, resp.), then $\mathbf{0}$ is a minimizer of f if and only if $\mathbf{0}$ is a minimizer (maximizer, resp.) of f_2 .

Proof. Since $f_2(\mathbf{0}) = 0$ and $\nabla f_2(\mathbf{0}) = \mathbf{0}$, the conclusion is clear. \square

6. Conclusions and further remarks

We proposed a computational and symbolic method to determine the type of a given isolated real critical point, which is degenerate, of a multivariate polynomial function. Given an isolation radius of the critical point, the tangency variety of the polynomial function at the critical point is used to define and compute its faithful radius. Elimination ideals and root isolation of univariate polynomials

¹ `RegularChains`: <http://www.regularchains.org/>.

are computed in finding a faithful radius. Once a faithful radius of the critical point is known, its type can be determined by isolating the real roots of a zero-dimensional polynomial system.

As we assume in this paper that the isolated critical point is the origin, we would like to remark that usually critical points are defined in some algebraic extensions. In this case, some difficulties may arise in implementing the algorithms proposed in this paper. In fact, many algebraic computations which can be done in the current computer algebra systems, like MAPLE, is more efficient or only available in the rational number field \mathbb{Q} . Although our theory works for all polynomials in $\mathbb{R}[X]$, as long as the critical point lies at the origin and the polynomial f is in $\mathbb{Q}[X]$, all computations can be efficiently done in the rational number field. If the critical point of interest, say c , is not the origin, we can use $\Gamma_{\mathbb{R}}(f, c)$ as defined in (4.1) instead of $\Gamma_{\mathbb{R}}(f)$. However, if either some coordinate of c is not rational or f is not in $\mathbb{Q}[X]$, then the defining polynomials of $\Gamma_{\mathbb{R}}(f, c)$ may not be in $\mathbb{Q}[X]$. Thus, the involved algebraic computations may not be efficient or not supported in the current computer algebra systems, like MAPLE.

Since the algorithms presented in this paper are based on algebraic computations of equidimensional decompositions, elimination ideals and solving zero-dimensional polynomial systems, the complexity in our algorithms depends heavily on these corresponding algorithms and we do not give details here. The case when the critical point of interest is non-isolated is also interesting and the extension of our method in the present paper to the non-isolated case will be studied in future work.

Acknowledgements

The authors are very grateful for the comments of two anonymous referees which helped to improve the presentation. The first author was supported by the Chinese National Natural Science Foundation under grants 11401074, 11571350. The second author was partially supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED), grant 101.04-2016.05.

References

- Alonso, M.-E., Becker, E., Roy, M.F., Wörmann, T., 1996. Zeros, multiplicities, and idempotents for zero-dimensional systems. In: González-Vega, L., Recio, T. (Eds.), *Algorithms in Algebraic Geometry and Applications*. Birkhäuser Basel, Basel, pp. 1–15.
- Aubry, P., Rouillier, F., Din, M.S.E., 2002. Real solving for positive dimensional systems. *J. Symb. Comput.* 34 (6), 543–560.
- Bank, B., Giusti, M., Heintz, J., Mbakop, G., 1997. Polar varieties, real equation solving, and data structures: the hypersurface case. *J. Complex.* 13 (1), 5–27.
- Bank, B., Giusti, M., Heintz, J., Safey El Din, M., Schost, E., 2010. On the geometry of polar varieties. *Appl. Algebra Eng. Commun. Comput.* 21 (1), 33–83.
- Barone-Netto, A., 1984. Jet-detectable extrema. *Proc. Am. Math. Soc.* 92 (4), 604–608.
- Barone-Netto, A., Gorni, G., Zampieri, G., 1996. Local extrema of analytic functions. *NoDEA Nonlinear Differ. Equ. Appl.* 3 (3), 287–303.
- Basu, S., Pollack, R., Roy, M.-F., 2006. *Algorithms in Real Algebraic Geometry*. Springer-Verlag, Berlin.
- Bochnak, J., Coste, M., Roy, M.-F., 1998. *Real Algebraic Geometry*. Springer, Berlin.
- Bolis, T.S., 1980. Degenerate critical points. *Math. Mag.* 53 (5), 294–299.
- Caviness, B.F., Johnson, J.R., 1998. Quantifier Elimination and Cylindrical Algebraic Decomposition. *Texts and Monographs in Symbolic Computation*. Springer-Verlag, Wien.
- Chen, C., Moreno Maza, M., 2014a. Cylindrical algebraic decomposition in the RegularChains library. In: Hong, H., Yap, C. (Eds.), *Mathematical Software – ICMS 2014*. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 425–433.
- Chen, C., Moreno Maza, M., 2014b. Quantifier elimination by cylindrical algebraic decomposition based on regular chains. In: *Proceedings of 2014 International Symposium on Symbolic and Algebraic Computation. ISSAC'14*. ACM, New York, NY, USA, pp. 91–98.
- Chen, C., Moreno Maza, M., 2014c. Real quantifier elimination in the RegularChains library. In: Hong, H., Yap, C. (Eds.), *Mathematical Software – ICMS 2014*. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 283–290.
- Chen, C., Moreno Maza, M., 2016. Quantifier elimination by cylindrical algebraic decomposition based on regular chains. In: *Special Issue on the Conference ISSAC 2014: Symbolic Computation and Computer Algebra*. *J. Symb. Comput.* 75, 74–93.
- Cox, D.A., Little, J., O’Shea, D., 2007. *Ideals, Varieties, and Algorithms – An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Graduate Texts in Mathematics. Springer-Verlag, New York.
- Cushing, J.M., 1975. Extremal tests for scalar functions of several real variables at degenerate critical points. *Aequ. Math.* 13, 89–96.
- Demazure, M., 2000. *Bifurcations and Catastrophes. Geometry of Solutions to Nonlinear Problems*. Universitext. Springer, Berlin.
- Durfee, A.H., 1998. The index of $grad f(x, y)$. *Topology* 37 (6), 1339–1361.
- Faugère, J., Gianni, P., Lazard, D., Mora, T., 1993. Efficient computation of zero-dimensional Gröbner bases by change of ordering. *J. Symb. Comput.* 16 (4), 329–344.

- Giusti, M., Heintz, J., Morais, J.E., Pardo, L.M., 1995. When polynomial equation systems can be “solved” fast? In: Cohen, G., Giusti, M., Mora, T. (Eds.), *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 205–231.
- Giusti, M., Lecerf, G., Salvy, B., 2001. A Gröbner free alternative for polynomial system solving. *J. Complex.* 17 (1), 154–211.
- Greuel, G.M., Pfister, G., 2002. *A Singular Introduction to Commutative Algebra*. Springer-Verlag, Berlin.
- Guillemin, V., Pollack, A., 1974. *Differential Topology*. Prentice Hall, New Jersey.
- Hà, H.V., Phạm, T.S., 2008. On the Lojasiewicz exponent at infinity of real polynomials. *Ann. Pol. Math.* 94, 197–208.
- Hà, H.V., Phạm, T.S., 2017. *Genericity in Polynomial Optimization*. World Scientific Publishing.
- Kunz, E., 1988. *Introduction to Commutative Algebra and Algebraic Geometry*. Springer, Berlin.
- Lasserre, J.B., 2009. *Moments, Positive Polynomials and Their Applications*. Imperial College Press, London, UK.
- Laurent, M., 2009. Sums of squares, moment matrices and optimization over polynomials. In: *Emerging Applications of Algebraic Geometry of IMA Volumes in Mathematics and Its Applications*, vol. 149. Springer, pp. 157–270.
- Lecerf, G., 2003. Computing the equidimensional decomposition of an algebraic closed set by means of lifting fibers. *J. Complex.* 19 (4), 564–596.
- Marshall, M., 2008. *Positive Polynomials and Sums of Squares*. *Mathematical Surveys and Monographs*, vol. 146. American Mathematical Society, Providence, RI.
- Milnor, J., 1968. *Singular Points of Complex Hypersurfaces*. *Annals of Mathematics Studies*, vol. 61. Princeton University Press, Princeton.
- Murty, K.G., Kabadi, S.N., 1987. Some NP-complete problems in quadratic and nonlinear programming. *Math. Program.* 39 (2), 117–129.
- Nesterov, Y., 2000. *Squared Functional Systems and Optimization Problems*. Springer US, Boston, MA, pp. 405–440.
- Nesterov, Y., 2003. *Random Walk in a Simplex and Quadratic Optimization over Convex Polytopes*. CORE Discussion Paper. UCL, Louvain-la-Neuve.
- Nie, J., 2015. The hierarchy of local minimums in polynomial optimization. *Math. Program.* 151 (2), 555–583.
- Nie, J., Ranestad, K., 2009. Algebraic degree of polynomial optimization. *SIAM J. Optim.* 20 (1), 485–502.
- Qi, L., 2004. Extrema of a real polynomial. *J. Glob. Optim.* 30 (4), 405–433.
- Rouillier, F., 1999. Solving zero-dimensional systems through the rational univariate representation. *Appl. Algebra Eng. Commun. Comput.* 9 (5), 433–461.
- Safey El Din, M., 2008. Computing the global optimum of a multivariate polynomial over the reals. In: *Proceedings of 2008 International Symposium on Symbolic and Algebraic Computation*. ISSAC'08. ACM, New York, NY, USA, pp. 71–78.
- Safey El Din, M., Schost, E., 2003. Polar varieties and computation of one point in each connected component of a smooth real algebraic set. In: *Proceedings of 2003 International Symposium on Symbolic and Algebraic Computation*. ISSAC'03. ACM, New York, NY, USA, pp. 224–231.
- Safey El Din, M., Schost, E., 2017. A nearly optimal algorithm for deciding connectivity queries in smooth and bounded real algebraic sets. *J. ACM* 63 (6), 1–48.
- Shafarevich, I.R., 1977. *Basic Algebraic Geometry 1*. Springer-Verlag.
- Xiao, S., Zeng, G., 2015. Equality-constrained minimization of polynomial functions. *Sci. China Math.* 58 (10), 2181–2204.